

THE AREA FORMULA FOR LIPSCHITZ MAPPINGS OF CARNOT–CARATHÉODORY SPACES¹

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Abstract

We prove the sub-Riemannian analog of the area formula for Lipschitz (in sub-Riemannian sense) mappings of equiregular Carnot–Carathéodory spaces.

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1 Introduction

In Euclidean analysis, the well-known area formula [1]

$$\int_U \mathcal{J}(\varphi, x) d\mathcal{H}^n(x) = \int_{\mathbb{R}^k} \sum_{x \in \varphi^{-1}(y)} \chi_U(x) d\mathcal{H}^n(y)$$

is proved for large classes of mappings $\varphi : U \rightarrow \mathbb{R}^k$, $U \subset \mathbb{R}^n$, $n \geq k$, possessing some regularity properties. Such classes include continuously differentiable mappings, Lipschitz mappings; Sobolev mappings and approximately differentiable mappings with Luzin property \mathcal{N} (see e. g. [2]), etc. This formula is generalized to wide classes of mappings of Riemannian manifolds and metric spaces [1, 3, 4, 5, 6, 7]. In many proofs of the area formula, the approximation of the initial mapping φ by the tangent one

$$w \mapsto D\varphi(x)[w]$$

is essentially used. In particular, the bi-Lipschitz equivalence of the metrics in the manifold and in the tangent space is applicable in such proofs.

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On equiregular Carnot–Carathéodory spaces (or simply Carnot manifolds) (see Definition 2.1 below), there are two structures, namely, Riemannian and *sub-Riemannian*. Moreover, metrics corresponding to these two structures, are not bi-Lipschitz equivalent. Thus, mappings Lipschitz with respect to sub-Riemannian metrics may not be Lipschitz with respect to Riemannian metrics, consequently they may not be differentiable in the classical sense on a set of non-zero exterior measure. For the mappings of Carnot manifolds, there exists the specific notion of the *sub-Riemannian differentiability*, or *hc-differentiability* [8] of mappings of Carnot manifolds.

Definition 1.1. A mapping $\varphi : U \rightarrow \tilde{\mathbb{M}}$, $U \subset \mathbb{M}$, where \mathbb{M} and $\tilde{\mathbb{M}}$ are Carnot manifolds, is *hc-differentiable* at a point $u \in U$ if there exists a horizontal homomorphism $L_u : (\mathcal{G}^u \mathbb{M}, d_{cc}^u) \rightarrow (\mathcal{G}^{\varphi(u)} \tilde{\mathbb{M}}, \tilde{d}_{cc}^{\varphi(u)})$ of local Carnot groups, such that

$$\tilde{d}_{cc}(\varphi(w), L_u[w]) = o(d_{cc}(u, w)) \text{ as } U \cap \mathcal{G}^u \mathbb{M} \ni w \rightarrow u.$$

Hereinafter, we denote L_u by $\hat{D}\varphi(u)$.

In particular cases when both Carnot manifolds are just Carnot groups, the notion of the *hc-differential* coincides with the one of the \mathcal{P} -differential introduced by P. Pansu [9]. This definition generalizes the classical definition of differentiability since a local Carnot group approximates the initial Carnot manifold with respect to a sub-Riemannian metric (just like a tangent space approximates a Riemannian manifold with respect to Riemannian metric). One of results of the paper [8] is the following:

Theorem 1.2. *Lipschitz (in the sub-Riemannian sense) mappings of Carnot manifolds are hc-differentiable almost everywhere.*

Theorem 1.3 ([8]). *Let $\varphi : \mathbb{M} \rightarrow \tilde{\mathbb{M}}$ be a contact C^1 -mapping of Carnot manifolds (in the Riemannian sense). Then, it is continuously hc-differentiable everywhere on \mathbb{M} (i. e., its hc-differential $\hat{D}\varphi(u)$ is continuous on $u \in \mathbb{M}$).*

Nevertheless, up to now, the problem on the area formula for Lipschitz mappings of Carnot manifolds has been solved only for some particular cases, i. e., for mappings of Carnot groups (a particular case of a Carnot manifold) [10, 11] and for classes of C^1 -smooth (in the classical sense) contact mappings of Carnot manifolds [12]. In [10], the author uses the approximation (with respect to sub-Riemannian metric) of the initial mapping by the “tangent” one defined via \mathcal{P} -differential. The main result of [10] is the following

Theorem 1.4 (see [10, Definition 2.20 and Theorem 3.3].). *Suppose that $\varphi : \mathbb{G} \rightarrow \tilde{\mathbb{G}}$ is a Lipschitz (with respect to sub-Riemannian metrics) map of two Carnot groups. Then, for any \mathcal{H}^ν -measurable set $E \subset \mathbb{G}$ (here ν is Hausdorff dimension of \mathbb{G}), we have*

$$\int_E \mathcal{J}(x) d\mathcal{H}^\nu(x) = \int_{\tilde{\mathbb{G}}} \sum_{x: x \in \varphi^{-1}(y)} \chi_E(y) d\mathcal{H}^\nu(y),$$

where Jacobian $\mathcal{J}(x)$ equals

$$\mathcal{J}(x) = \lim_{t \rightarrow 0} \left\{ \frac{\mathcal{H}^\nu(\varphi(B_{cc}(y, t)))}{\mathcal{H}^\nu(B_{cc}(y, t))} \middle| y \in B_{cc}(x, t) \right\}, \quad (1.1)$$

where Hausdorff measures are constructed with respect to d_{cc} .

In [11], the main idea is to use the local \mathcal{H}^ν -measure distortion under φ as a Jacobian:

Theorem 1.5 (see [11, Definition 10 and Theorem 4.4].). *Suppose that $\varphi : \mathbb{G} \rightarrow \tilde{\mathbb{G}}$ is a Lipschitz (with respect to sub-Riemannian metrics) map of two Carnot groups. Then, for any \mathcal{H}^ν -measurable set $E \subset \mathbb{G}$ (here ν is Hausdorff dimension of \mathbb{G}), we have*

$$\int_E \mathcal{J}_\nu(\hat{D}\varphi(x)) d\mathcal{H}^\nu(x) = \int_{\tilde{\mathbb{G}}} \sum_{x: x \in \varphi^{-1}(y)} \chi_E(y) d\mathcal{H}^\nu(y),$$

where Jacobian $\mathcal{J}_\nu(\hat{D}\varphi(x))$ equals

$$\mathcal{J}_\nu(\hat{D}\varphi(x)) = \frac{\mathcal{H}^\nu(\hat{D}\varphi(y)[B_{cc}(0, 1)])}{\mathcal{H}^\nu(B_{cc}(0, 1))}, \quad (1.2)$$

where Hausdorff measures are constructed with respect to d_{cc} .

Finally, in [12], the sub-Riemannian area formula is derived via the Riemannian one. The result is

Theorem 1.6 (The Area Formula for Smooth Mappings [12]). *Let $\varphi : \mathbb{M} \rightarrow \tilde{\mathbb{M}}$ be a contact C^1 -mapping. Then the area formula*

$$\int_{\mathbb{M}} f(x) \mathcal{J}^{SR}(\varphi, x) d\mathcal{H}^\nu(x) = \int_{\tilde{\mathbb{M}}} \sum_{x: x \in \varphi^{-1}(y) \cap U} f(x) d\mathcal{H}^\nu(y),$$

where $f : \mathbb{M} \rightarrow \mathbb{E}$ (here \mathbb{E} is an arbitrary Banach space) is such that the function $f(x)\mathcal{J}^{SR}(\varphi, x)$ is integrable, and

$$\mathcal{J}^{SR}(\varphi, x) = \sqrt{\det(\widehat{D}\varphi(x)^*\widehat{D}\varphi(x))} \quad (1.3)$$

is the sub-Riemannian Jacobian of φ at x , is valid. Here the Hausdorff measures are constructed with respect to metrics d_2 and \widetilde{d}_2 with the multiple ω_ν .

Note that the definition (1.3) of the sub-Riemannian Jacobian (that is, its analytic expression via the values of the hc -differential) is new even for mappings of Carnot groups.

On the one hand, in view of non-differentiability of Lipschitz in the sub-Riemannian sense mappings, it is impossible to derive the sub-Riemannian area formula for *arbitrary* Lipschitz mappings of Carnot manifolds via the Riemannian one. On the other hand, the hc -differential $\widehat{D}\varphi$ of a mapping $\varphi : \mathbb{M} \rightarrow \widetilde{M}$ at arbitrary point u acts on local Carnot groups $\mathcal{G}^u\mathbb{M}$ and $\mathcal{G}^{\varphi(u)}\widetilde{M}$, in which the sub-Riemannian metrics are not equivalent to the ones in a Carnot manifold [15], thus the relation $d_{\mathbb{M}}(\varphi(w), \varphi(v)) = (1 + o(1))\widehat{d}_{\mathcal{G}^{\varphi(u)}}(\widehat{D}\varphi(u)[v], \widehat{D}\varphi(u)[w])$, where $o(1) \rightarrow 0$ as $v, w \rightarrow u$, near the point u cannot be obtained.

In this paper, we give a new approach to investigation of Lipschitz mappings of Carnot manifolds based on its hc -differentiability only, and its “partial” approximation by a “tangent” mapping. Such approach is new even for mappings of Euclidean spaces. We prove the area formula for Lipschitz mappings of Carnot manifolds (see also Theorem 3.12 below):

Theorem 1.7. *Suppose that $D \subset \mathbb{M}$ is a measurable set, and the mapping $\varphi : D \rightarrow \widetilde{M}$ is Lipschitz with respect to sub-Riemannian quasimetrics d_2 and \widetilde{d}_2 . Then the area formula*

$$\int_D f(x)\mathcal{J}^{SR}(\varphi, x) d\mathcal{H}^\nu(x) = \int_{\varphi(D)} \sum_{x: x \in \varphi^{-1}(y)} f(x) d\mathcal{H}^\nu(y), \quad (1.4)$$

where $f : D \rightarrow \mathbb{E}$ (here \mathbb{E} is an arbitrary Banach space) is such that the function $f(x)\mathcal{J}^{SR}(\varphi, x)$ is integrable, and the sub-Riemannian Jacobian is the same as in (1.3), is valid. Here the Hausdorff measures are constructed with respect to metrics d_2 and \widetilde{d}_2 with the multiple ω_ν .

Remark 1.8. Note that (see, e. g., [10]) that the definitions (1.1) and (1.2) are equivalent. Next, it is easy to see that Theorems 1.4 and 1.5 are particular cases of Theorem 1.7. Indeed, in view of Ball–Box Theorem [13], [14], Hausdorff measures constructed with respect to Carnot–Carathéodory

metric d_{cc} (see Definition 2.6) and with respect to the quasimetric d_2 (see Definition 2.7), are absolutely continuous one with respect to another. Since on a Carnot group these measures are left-invariant, then the derivative of one with respect to another is constant. Denote it by $\mathcal{D}_{2,cc}$ in the preimage and by $\widehat{\mathcal{D}}_{2,cc}$ in the image. In view of the validity of (1.4) for the mapping $\psi(y) = \widehat{D}\varphi(x)[y] : B_{cc}(0, 1) \rightarrow \widetilde{\mathbb{G}}$ we infer

$$\mathcal{J}^{SR}(\varphi, x) = \frac{\mathcal{H}^\nu(\widehat{D}\varphi(y)[B_{cc}(0, 1)])}{\mathcal{H}^\nu(B_{cc}(0, 1))},$$

where Hausdorff measures are constructed with respect to d_2 . Thus, on the one hand,

$$\begin{aligned} \int_D \mathcal{J}^{SR}(\varphi, x) d\mathcal{H}^\nu(x) &= \int_D \frac{\widehat{\mathcal{D}}_{2,cc} \mathcal{H}_{cc}^\nu(\widehat{D}\varphi(y)[B_{cc}(0, 1)])}{\mathcal{D}_{2,cc} \mathcal{H}_{cc}^\nu(B_{cc}(0, 1))} d\mathcal{H}^\nu(x) \\ &= \int_D \frac{\widehat{\mathcal{D}}_{2,cc} \mathcal{H}_{cc}^\nu(\widehat{D}\varphi(y)[B_{cc}(0, 1)])}{\mathcal{D}_{2,cc} \mathcal{H}_{cc}^\nu(B_{cc}(0, 1))} \mathcal{D}_{2,cc} d\mathcal{H}_{cc}^\nu(x) \\ &= \int_D \mathcal{D}_{2,cc} J_\nu(\widehat{D}\varphi(x)) d\mathcal{H}_{cc}^\nu(x) = \mathcal{D}_{2,cc} \int_D J_\nu(\widehat{D}\varphi(x)) d\mathcal{H}_{cc}^\nu(x), \end{aligned}$$

where Hausdorff measures \mathcal{H}_{cc}^ν are constructed with respect to d_{cc} 's. On the other hand,

$$\begin{aligned} \int_{\varphi(D)} \sum_{x: x \in \varphi^{-1}(y)} \chi_D(y) d\mathcal{H}^\nu(y) &= \int_{\varphi(D)} \sum_{x: x \in \varphi^{-1}(y)} \chi_D(y) \mathcal{D}_{2,cc} d\mathcal{H}_{cc}^\nu(y) \\ &= \mathcal{D}_{2,cc} \int_{\varphi(D)} \sum_{x: x \in \varphi^{-1}(y)} \chi_D(y) d\mathcal{H}_{cc}^\nu(y). \end{aligned}$$

Since the value $\mathcal{D}_{2,cc}$ is strictly positive, Theorems 1.5 and 1.4 follow from Theorem 1.7.

Emphasize here that although its proof uses in one of its steps the sub-Riemannian area formula for C^1 -smooth (in the classical sense) mappings, this area formula is not its direct consequence, and its proof requires approaches and methods that are essentially new in comparison with the classical situation of obtaining a result for Lipschitz mappings via the same results for C^1 -mappings.

Theorem 1.6 and the classical area formula for mappings of Riemannian manifolds are particular cases of Theorem 1.7. Moreover, Theorems 1.4

and 1.5 can also be considered as consequences of Theorem 1.7. The difference is in definition of Jacobians: in [10], [11], the definition of Jacobian uses measure of balls in Carnot–Carathéodory metrics and of their images under φ and $\widehat{D}\varphi$. The problem is that the measures of these images cannot be calculated since the structure of Carnot–Carathéodory balls is unknown in general case. In Theorems 1.6 and 1.7, a sub-Riemannian quasimetric is used. Its advantage is that the structure of balls in this quasimetric is well-understandable, and they are easy to work with during the investigation on Jacobian and image properties. Moreover, it allows to write the exact analytic expression of the Jacobian. It is very important for application such as studying extremal surfaces on non-holonomic structures and many others.

2 Preliminaries

In this section, we introduce necessary definitions and mention important facts that we will need to prove the main result.

Definition 2.1 (compare with [16, 17]). Fix a connected Riemannian C^∞ -manifold \mathbb{M} of a topological dimension N . The manifold \mathbb{M} is called a *Carnot–Carathéodory space* if, in the tangent bundle $T\mathbb{M}$, there exists a filtration

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \dots \subsetneq H_i\mathbb{M} \subsetneq \dots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

of subbundles of the tangent bundle $T\mathbb{M}$ such that, for each point $p \in \mathbb{M}$, there exists a neighborhood $U \subset \mathbb{M}$ with a collection of C^1 -smooth vector fields X_1, \dots, X_N on it enjoying the following two properties. For each $v \in U$ we have

(1) $H_i\mathbb{M}(v) = H_i(v) = \text{span}\{X_1(v), \dots, X_{\dim H_i}(v)\}$ is a subspace of $T_v\mathbb{M}$ of a constant dimension $\dim H_i$, $i = 1, \dots, M$;

(2)

$$[X_i, X_j](v) = \sum_{\deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v) \quad (2.1)$$

where the *degree* $\deg X_k$ equals $\min\{m \mid X_k \in H_m\}$;

If, additionally, the third condition holds then the Carnot–Carathéodory space will be called the *Carnot manifold*:

(3) a quotient mapping $[\cdot, \cdot]_0 : H_1 \times H_j / H_{j-1} \mapsto H_{j+1} / H_j$ induced by Lie brackets is an epimorphism for all $1 \leq j < M$.

The subbundle $H\mathbb{M}$ is called *horizontal*.

The number M is called the *depth* of the manifold \mathbb{M} .

Definition 2.2. Consider Cauchy problem

$$\begin{cases} \dot{\gamma}(t) = \sum_{i=1}^N y_i X_i(\gamma(t)), & t \in [0, 1] \\ \gamma(0) = x, \end{cases}$$

where the vector fields X_1, \dots, X_N are C^1 -smooth. Then, for the point $y = \gamma(1)$ we write $y = \exp\left(\sum_{i=1}^N y_i X_i\right)(x)$.

The mapping $(y_1, \dots, y_N) \mapsto \exp\left(\sum_{i=1}^N y_i X_i\right)(x)$ is called *exponential*.

Definition 2.3. Suppose that $u \in \mathbb{M}$ and $(v_1, \dots, v_N) \in B_E(0, r)$, where $B_E(0, r)$ is a Euclidean ball in \mathbb{R}^N . Define a mapping $\theta_u(v_1, \dots, v_N) : B_E(0, r) \rightarrow \mathbb{M}$ as follows:

$$\theta_u(v_1, \dots, v_N) = \exp\left(\sum_{i=1}^N v_i X_i\right)(u).$$

It is known, that θ_u is a C^1 -diffeomorphism if $0 < r \leq r_u$ for some $r_u > 0$. The collection $\{v_i\}_{i=1}^N$ is called *the normal coordinates* or *the coordinates of the 1st kind* (with respect to $u \in \mathbb{M}$) of the point $v = \theta_u(v_1, \dots, v_N)$.

Assumption 2.4. Hereinafter, we consider points from a compactly embedded neighborhood $\mathcal{U} \Subset \mathbb{M}$ such that $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ for all $u \in \mathcal{U}$.

Definition 2.5. An absolutely continuous curve $\gamma : [0, 1] \rightarrow \mathbb{M}$ is called *horizontal* if $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$ for almost all $t \in [0, 1]$.

Definition 2.6. Carnot–Carathéodory distance d_{cc} between $x, y \in \mathbb{M}$ equals

$$d_{cc}(x, y) = \inf_{\gamma} l(\gamma),$$

where γ is a horizontal curve with endpoints x and y .

Now, we introduce the sub-Riemannian quasimetric locally equivalent to d_{cc} [18] which simplifies computations in the main theorems.

Definition 2.7. Let \mathbb{M} be a Carnot manifold of the topological dimension N and of the depth M , and suppose that $x = \exp\left(\sum_{i=1}^N x_i X_i\right)(u)$. The quasidistance $d_2(x, g)$ is defined as follows:

$$d_2(x, u) = \max \left\{ \left(\sum_{j=1}^{\dim H_1} |x_j|^2 \right)^{\frac{1}{2}}, \left(\sum_{j=\dim H_1+1}^{\dim H_2} |x_j|^2 \right)^{\frac{1}{2 \cdot \deg X_{\dim H_2}}}, \right. \\ \left. \dots, \left(\sum_{j=\dim H_{M-1}+1}^N |x_j|^2 \right)^{\frac{1}{2 \cdot \deg X_N}} \right\}.$$

Remark 2.8. The preimage of a ball $\text{Box}_2(u, r) = \{x \in \mathbb{M}_1 : d_2(x, u) < r\}$ in the quasimetric d_2 under the mapping θ_u equals $\text{Box}_2(0, r) = B_E^{n_1}(0, r) \times B_E^{n_2}(0, r^2) \times \dots \times B_E^{n_M}(0, r^M)$, where $B_E^{n_i}$, $i = 1, \dots, M$, are Euclidean balls of the dimensions $n_i = \dim H_i - \dim H_{i-1}$.

Such quasimetric is much more easier to deal with than the well known d_∞ , where $d_\infty(x, u) = \max_{i=1, \dots, N} \{|x_i|^{\frac{1}{\deg X_i}}\}$. The point is that in the case of d_∞ , the asymptotical shape of the section of a ball in d_∞ by a plane cannot be defined easily since any cube has several sections of different shapes. Since any section of a (Euclidean) ball is just a ball of lower dimension, it is convenient to consider sections of their Cartesian product, i. e., a ball in d_2 .

Property 2.9. It is easy to see that [19, 20, 18] the Hausdorff dimension of \mathbb{M} with respect to d_2 is equal to $\sum_{i=1}^M i(\dim H_i - \dim H_{i-1})$, where $\dim H_0 = 0$.

Theorem 2.10 ([18]). *Fix $u \in \mathbb{M}$. The coefficients*

$$\bar{c}_{ijk} = \begin{cases} c_{ijk}(u) \text{ of (2.1)} & \text{if } \deg X_i + \deg X_j = \deg X_k \\ 0, & \text{otherwise} \end{cases}$$

define a graded nilpotent Lie algebra.

We construct the Lie algebra \mathfrak{g}^u from Theorem 2.10 as a graded nilpotent Lie algebra of vector fields $\{(\hat{X}_i^u)'\}_{i=1}^N$ on \mathbb{R}^N such that the exponential mapping $(x_1, \dots, x_N) \mapsto \exp\left(\sum_{i=1}^N x_i (\hat{X}_i^u)'\right)(0)$ equals identity [21, 22]. In view of results of [23], the value of $(\hat{X}_j^u)'(0)$ is equal to a standard vector e_{i_j} , where $i_j \neq i_k$ if $j \neq k$, $j = 1, \dots, N$. We associate to each vector field from the obtained collection such a number i that $(\theta_u)_* \langle (\hat{X}_i^u)' \rangle(u) = X_i(u)$. By the construction, the relation

$$[(\hat{X}_i^u)', (\hat{X}_j^u)'] = \sum_{\deg X_k = \deg X_i + \deg X_j} c_{ijk}(u) (\hat{X}_k^u)' \quad (2.2)$$

holds for the vector fields $\{(\hat{X}_i^u)'\}_{i=1}^N$ everywhere on \mathbb{R}^N .

Notation 2.11. We use the following standard notations: for each N -dimensional multi-index $\mu = (\mu_1, \dots, \mu_N)$, its *homogeneous norm* equals $|\mu|_h = \sum_{i=1}^N \mu_i \deg X_i$, and $x^\mu = \prod_{i=1}^N x_i^{\mu_i}$ if $x = (x_1, \dots, x_N)$

Definition 2.12. The graded nilpotent Carnot group $\mathbb{G}_u\mathbb{M}$ corresponding to the Lie algebra \mathfrak{g}^u , is called the *nilpotent tangent cone* of \mathbb{M} at $u \in \mathbb{M}$. We construct $\mathbb{G}_u\mathbb{M}$ in \mathbb{R}^N as a groupalgebra [21], that is, the exponential map is identical:

$$\exp\left(\sum_{i=1}^N x_i(\widehat{X}_i^u)'\right)(0) = (x_1, \dots, x_N).$$

By Campbell–Hausdorff formula, the group operation is defined for the basis vector fields $(\widehat{X}_i^u)'$ on \mathbb{R}^N , $i = 1, \dots, N$, to be left-invariant [21]: if

$$x = \exp\left(\sum_{i=1}^N x_i(\widehat{X}_i^u)'\right), \quad y = \exp\left(\sum_{i=1}^N y_i(\widehat{X}_i^u)'\right)$$

then

$$x \cdot y = z = \exp\left(\sum_{i=1}^N z_i(\widehat{X}_i^u)'\right),$$

where

$$z_i = x_i + y_i + \sum_{\substack{|\mu+\beta|_h = \deg X_i, \\ \mu, \beta > 0}} F_{\mu, \beta}^i(u) x^\mu y^\beta.$$

Property 2.13. It is easy to see that $\widehat{X}_i^u(u) = X_i(u)$, $i = 1, \dots, N$.

Definition 2.14. For $u, g \in \mathbb{M}$, define the exponential mapping

$$\widehat{\theta}_g^u(x_1, \dots, x_N) : B_E(0, r) \rightarrow \mathbb{M}$$

as $\widehat{\theta}_g^u(x_1, \dots, x_N) = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^u\right)(g)$, which is a C^1 -diffeomorphism for all $0 < r \leq r_{u,g}$ for some $r_{u,g} > 0$.

Assumption 2.15. We suppose that the neighborhood under consideration \mathcal{U} is such that $\mathcal{U} \subset \widehat{\theta}_g^u(B_E(0, r_{u,g}))$ for all $u, g \in \mathcal{U}$.

Notation 2.16. The quasimetric d_2^u with respect to the vector fields $\{\widehat{X}_i^u\}$ is defined similarly to the initial d_2 (defined with respect to $\{X_i\}$). A ball in d_2^u centered at x of a radius $r > 0$ is denoted by $\text{Box}_2^u(x, r)$.

Notation 2.17. We let the topological dimension of the manifold \mathbb{M} ($\widetilde{\mathbb{M}}$) be equal N (\widetilde{N}), and we let the Hausdorff dimension with respect to d_2 (\widetilde{d}_2) be

equal ν ($\tilde{\nu}$). The tangent spaces represented as the direct sums of quotient vector spaces

$$T_v \mathbb{M} = \bigoplus_{j=1}^M (H_j(v)/H_{j-1}(v)), \quad H_0 = \{0\},$$

$$\text{and } T_u \tilde{\mathbb{M}} = \bigoplus_{j=1}^{\tilde{M}} (\tilde{H}_j(u)/\tilde{H}_{j-1}(u)), \quad \tilde{H}_0 = \{0\},$$

at points $v \in \mathbb{M}$ and $u \in \tilde{\mathbb{M}}$, where $H_1 \subset T\mathbb{M}$ and $\tilde{H}_1 \subset T\tilde{\mathbb{M}}$ are corresponding *horizontal* subbundles, have structures of nilpotent graded Lie algebras [20]. Denote the dimensions of H_j/H_{j-1} ($\tilde{H}_j/\tilde{H}_{j-1}$) by symbols n_j (\tilde{n}_j), $j = 1, \dots, M$ (\tilde{M}).

Notation 2.18. Hereinafter, we denote the quasimetric d_2 in the preimage by the symbol d_2 , and we denote the quasimetric d_2 in the image by the symbol \tilde{d}_2 .

Assumption 2.19. We suppose that

- 1) a mapping φ is defined on a measurable set $D \subset \mathbb{M}$;
- 2) $\dim H_1 \leq \dim \tilde{H}_1$;
- 3) the basis vector fields in the preimage and in the image are $C^{1,\alpha}$ -smooth, $\alpha > 0$, and φ is Lipschitz with respect to d_2 and \tilde{d}_2 ($\tilde{d}_2(\varphi(u), \varphi(v)) \leq Ld_2(u, v)$ for all $u, v \in D$ and some $L < \infty$).

3 The Main Result

Theorem 3.1 ([8]). *Suppose that $D \subset \mathbb{M}$ is a measurable set, and let $\varphi : \mathbb{M} \rightarrow \tilde{\mathbb{M}}$ be a Lipschitz with respect to sub-Riemannian metrics mapping. Then, it is *hc-differentiable* almost everywhere. Namely, there exists a horizontal homomorphism $L_u : (\mathcal{G}^u \mathbb{M}, d_2^u) \rightarrow (\mathcal{G}^{\varphi(u)} \tilde{\mathbb{M}}, d_2^{\varphi(u)})$ of local Carnot groups, such that*

$$\tilde{d}_2(\varphi(w), L_u[w]) = o(d_2(u, w)) \text{ as } D \cap \mathcal{G}^u \mathbb{M} \ni w \rightarrow u.$$

Definition 3.2. The horizontal homomorphism $L_u : (\mathcal{G}^u \mathbb{M}, d_2^u) \rightarrow (\mathcal{G}^{\varphi(u)} \tilde{\mathbb{M}}, d_2^{\varphi(u)})$ is called the *hc-differential* of φ at u .

Corollary 3.3 ([8]). *Let $\varphi : \mathbb{M} \rightarrow \tilde{\mathbb{M}}$ be a contact (i. e., $D\varphi\langle H \rangle \subset \tilde{H}$) C^1 -mapping of Carnot manifolds (in the Riemannian sense). Then, it is continuously *hc-differentiable* everywhere on \mathbb{M} .*

Remark 3.4. Using the exponential mapping θ_u , we can consider L_u both as a homomorphism of local Carnot groups, and as a homomorphism of Lie algebras of these local Carnot groups.

Theorem 3.5 (Local Approximation Theorem [18, 13, 14]). *Suppose that $u, w, v \in \mathcal{U}$, and $d_2(u, w) = O(\varepsilon)$ and $d_2(u, v) = O(\varepsilon)$. Then we have*

$$|d_2(w, v) - d_2^u(w, v)| = O(\varepsilon^{1+\frac{1}{M}}),$$

where $O(1)$ is uniform on \mathcal{U} .

Remark that although the quasimetric in Theorem 3.5 is different from the one in [18], [13] and [14] the statement is the same since the scheme of the proof is the same.

Notation 3.6. Denote the hc -differential of φ at u by the symbol $\widehat{D}\varphi(u)$. Put $Z = \{u \in \mathbb{M} : \text{rank}(\widehat{D}\varphi(u)) < N\}$.

Remark 3.7. Given at least one point $u \in \mathbb{M}$ possessing the property $\text{rank } \widehat{D}\varphi(u) = N$, the item 2 of Assumption 2.19 implies

$$\dim H_i - \dim H_{i-1} \leq \dim \widetilde{H}_i - \dim \widetilde{H}_{i-1}, \quad i = 1, \dots, M,$$

where $\dim H_0 = 0$ and $\dim \widetilde{H}_0 = 0$. Indeed, it is enough to take into account the property

$$\widehat{D}\varphi(u)[X, Y] = [\widehat{D}\varphi(u)X, \widehat{D}\varphi(u)Y],$$

where X, Y are vector fields corresponding to the local Carnot group $\mathcal{G}^u\mathbb{M}$, the properties of the local Carnot group [20], and property 4 from Definition 2.1.

Definition 3.8. The (spherical) Hausdorff \mathcal{H}^ν -measure of a set $E \subset \varphi(\mathbb{M})$ is defined as

$$\mathcal{H}^\nu(E) = \omega_\nu \liminf_{\delta \rightarrow 0} \left\{ \sum_{i \in \mathbb{N}} r_i^\nu : \bigcup_{i \in \mathbb{N}} \text{Box}_2(x_i, r_i) \supset E, x_i \in E, r_i \leq \delta \right\}.$$

Definition 3.9 ([12]). The *sub-Riemannian Jacobian* equals

$$\mathcal{J}^{SR}(\varphi, x) = \sqrt{\det(\widehat{D}\varphi(x)^* \widehat{D}\varphi(x))}.$$

Theorem 3.10. *We have $\mathcal{H}^\nu(\varphi(Z)) = 0$, where*

$$Z = \{x \in \mathbb{M} : \text{rank } \widehat{D}\varphi(x) < N\}.$$

Proof. The proof is based on a sharp modification of the arguments given in [24].

Note that $\tilde{d}_2^{\varphi(y)}(\varphi(z), \widehat{D}\varphi(y)[z]) = o(d_2(y, z))$. By another words, if $y \in Z$, then the image of $\text{Box}_2(y, t) = \text{Box}_2^y(y, t)$ is a subset of $o(t)$ -neighborhood (with respect to $\tilde{d}_2^{\varphi(y)}$) of the image of $\mathcal{G}^y\mathbb{M} \cap \text{Box}_2^y(y, t)$ under $\widehat{D}\varphi(y)$.

Since at y we have $\text{rank } \widehat{D}\varphi(y) < N$, then, the Hausdorff dimension (with respect to $\tilde{d}_2^{\varphi(y)}$) of $\widehat{D}\varphi(y)[\mathcal{G}^y\mathbb{M} \cap \text{Box}_2^y(y, t)]$ does not exceed $\nu - 1$. Indeed, taking into account the property 3 in Definition 2.1, we have for any basis vector field $X_m \in H_j$, $j > 1$, that there exist vector fields $X \in H_{j-1}$ and $Y \in H_1$ such that $X_m = [X, Y]$. Next, $[X, Y](v) = \sum_{l: X_{k_l} \in H_{j-1}, Y_{q_l} \in H_1} a_l(v)[X_{k_l}, Y_{q_l}](v) + Z$, where $Z \in H_{j-1}$. By the second property, $\sum_{l: X_{k_l} \in H_{j-1}, Y_{q_l} \in H_1} a_l(v)[X_{k_l}, Y_{q_l}](v) = \sum_{l: X_{k_l} \in H_{j-1}, Y_{q_l} \in H_1} \sum_{p: \deg X_p \leq j} a_l(v)c_{k_l q_l p}(v)X_p(v)$. By the property of vector fields of the local Carnot group,

$$\sum_{l: X_{k_l} \in H_{j-1}, Y_{q_l} \in H_1} a_l(v)[\widehat{X}_{k_l}^y, \widehat{Y}_{q_l}^y](v) = \sum_{l: X_{k_l} \in H_{j-1}, Y_{q_l} \in H_1} \sum_{p: \deg X_p = j} a_l(v)c_{k_l q_l p}(y)\widehat{X}_p(v),$$

and, by assumption $X_m = [X, Y]$ we have that the last sum equals \widehat{X}_m^u . In view of the property

$$\widehat{D}\varphi(y)[X, Y] = [\widehat{D}\varphi(y)X, \widehat{D}\varphi(y)Y],$$

where X, Y are vector fields corresponding to the local Carnot group $\mathcal{G}^y\mathbb{M}$, we infer that the sum of degrees of the images under $\widehat{D}\varphi(y)$ of the basis vector fields cannot be bigger than ν ; moreover, it equals ν only if $\text{rank } \widehat{D}\varphi(y) = N$. In all the other cases, this sum does not exceed $\nu - 1$.

For $0 < \sigma < \infty$, take $\varepsilon > 0$, and suppose without loss of generality that Z is compact, and that both values $o(1)$ in the definition of hc -differentiability and in Local Approximation Theorem 3.5 do not exceed ε . Fix $\delta > 0$ and construct the covering of Z by balls $\{\text{Box}_2(y_i, t_i)\}_{i \in \mathbb{N}, y_i \in Z, t_i \leq \delta}$, from the definition of \mathcal{H}'_δ , such that

$$\omega_\nu \sum_{i \in \mathbb{N}} t_i^\nu \leq \mathcal{H}^\nu(Z) + \sigma.$$

Fix $i \in \mathbb{N}$ and estimate $\mathcal{H}^\nu(\varphi(\text{Box}_2(y_i, t_i)))$. The image $\varphi(\text{Box}_2(y_i, t_i))$ is a subset of a εt_i -neighborhood of $\widehat{D}\varphi(y_i)[\text{Box}_2^{y_i}(y_i, t_i)]$ which has sub-Riemannian Hausdorff dimension ν_i not exceeding $\nu - 1$. Consider the family of balls

$$\{\text{Box}_2^{\varphi(y_i)}(s, 2\varepsilon t_i)\}_{s \in \widehat{D}\varphi(y_i)[\text{Box}_2^{y_i}(y_i, t_i)]}$$

in $\mathcal{G}^{\varphi(y_i)}\widetilde{\mathbb{M}}$ with centers on the set $\widehat{D}\varphi(y_i)[\text{Box}_2^{y_i}(y_i, t_i)]$ and of radii $2\varepsilon t_i$, which covers the set $\varphi(\text{Box}_2(y_i, t_i))$. In view of the degeneracy of $\widehat{D}\varphi(y_i)$, we have that the volume of the intersection

$$\text{Box}_2^{\varphi(y_i)}(s, 2\varepsilon t_i) \cap \widehat{D}\varphi(y_i)[\text{Box}_2^{y_i}(y_i, t_i)]$$

is not less than $O((\varepsilon t_i)^{\nu_i})$, where $O(1)$ is strictly greater than zero uniformly on some compact neighborhood (here we also take into account the left-invariance on $\mathcal{G}^{\varphi(y_i)}\widetilde{\mathbb{M}}$, and we suppose without loss of generality that Z is a subset of such compact neighborhood). By Lipschitzity of φ and degeneracy of $\widehat{D}\varphi(y_i)$, we have that the volume of $\widehat{D}\varphi(y_i)[\text{Box}_2^{y_i}(y_i, t_i)]$ does not exceed $O(t_i^{\nu_i})$ (here $O(1)$ is also uniform on the compact neighborhood under consideration). Here $\nu_i \leq \nu - 1$ depends on the degeneracy of $\widehat{D}\varphi$ at y_i , namely, it equals sum of degrees of all the basis vector fields in $\mathcal{G}^u\mathbb{M}$ on which $\widehat{D}\varphi(y_i)$ is non-degenerate, and images of which are independent.

Since in the local Carnot group $\mathcal{G}^{\varphi(y_i)}\widetilde{\mathbb{M}}$ the quasimetric $\widetilde{d}_2^{\varphi(y_i)}$ is locally equivalent to Carnot–Carathéodory metric $\widetilde{d}_{cc}^{\varphi(y_i)}$, then we obtain applying $5r$ -Covering Lemma that there exist not more than

$$\frac{O(t_i^{\nu_i})}{O((\varepsilon t_i)^{\nu_i})} = \frac{1}{O(\varepsilon^{\nu-1})}$$

of balls $\{\text{Box}_2^{\varphi(y_i)}(s_j, r_j)\}_{j \in \mathbb{N}}$ covering $\varphi(\text{Box}_2(y_i, t_i))$, the radii of which vary from $2\varepsilon t_i$ to $l \cdot 2\varepsilon t_i$, and such that the corresponding balls $\{\text{Box}_2^{\varphi(y_i)}(s_j, 2\varepsilon t_i)\}_{j \in \mathbb{N}}$ are disjoint. Here the constant l depends on the equivalence coefficients of $\widetilde{d}_2^{\varphi(y_i)}$ and $\widetilde{d}_{cc}^{\varphi(y_i)}$, and of $5r$ -Covering Lemma [1], and $O(1)$ is strictly greater than zero uniformly in $i \in \mathbb{N}$.

In view of Local Approximation Theorem 3.5 for \widetilde{d}_2 and $\widetilde{d}_2^{\varphi(y_i)}$ (we may assume without loss of generality that on the set $\varphi(\text{Box}_2(y_i, t_i))$ we have $|\widetilde{d}_2 - \widetilde{d}_2^{\varphi(y_i)}| \leq \varepsilon t_i$), the collection of the balls $\{\text{Box}_2(s_j, 2r_j)\}_{j \in \mathbb{N}}$ covers the set $\varphi(\text{Box}_2(y_i, t_i))$. Consequently,

$$\mathcal{H}_{4l\varepsilon\delta}^\nu(\varphi(\text{Box}_2(y_i, t_i))) \leq (4l\varepsilon t_i)^\nu \cdot \frac{1}{O(\varepsilon^{\nu-1})} = O(\varepsilon) \cdot t_i^\nu,$$

where $O(\cdot)$ is uniform on $\varphi(Z)$. Thus,

$$\mathcal{H}_{4l\varepsilon\delta}^\nu(\varphi(Z)) \leq \mathcal{H}_{4l\varepsilon\delta}^\nu\left(\bigcup_{i \in \mathbb{N}} \varphi(\text{Box}_2(y_i, t_i))\right) \leq O(\varepsilon) \sum_{i \in \mathbb{N}} t_i^\nu \leq O(\varepsilon)(\mathcal{H}^\nu(Z) + \sigma).$$

Here $O(1)$ is uniform in all $\delta > 0$ small enough. If $\delta \rightarrow 0$ then we have $\varepsilon \rightarrow 0$, and the theorem follows. \square

Theorem 3.11 (The Area Formula for Smooth Mappings [12]). *Let $\varphi : \mathbb{M} \rightarrow \tilde{\mathbb{M}}$ be a contact C^1 -mapping which is continuously hc -differentiable (i. e., its hc -differential $\widehat{D}\varphi(u)$ is continuous on $u \in \mathbb{M}$) everywhere. Then the area formula*

$$\int_{\mathbb{M}} f(x) \sqrt{\det(\widehat{D}\varphi(x)^* \widehat{D}\varphi(x))} d\mathcal{H}^\nu(x) = \int_{\tilde{\mathbb{M}}} \sum_{x: x \in \varphi^{-1}(y)} f(x) d\mathcal{H}^\nu(y),$$

where $f : \mathbb{M} \rightarrow \mathbb{E}$ (here \mathbb{E} is an arbitrary Banach space) is such that the function $f(x) \sqrt{\det(\widehat{D}\varphi(x)^* \widehat{D}\varphi(x))}$ is integrable, is valid. Here the Hausdorff measures are constructed with respect to metrics d_2 and \tilde{d}_2 with the multiple ω_ν .

Theorem 3.12. *Suppose that $D \subset \mathbb{M}$ is a measurable set, and the mapping $\varphi : D \rightarrow \tilde{\mathbb{M}}$ is Lipschitz with respect to sub-Riemannian quasimetrics d_2 and \tilde{d}_2 . Then the area formula*

$$\int_D f(x) \sqrt{\det(\widehat{D}\varphi(x)^* \widehat{D}\varphi(x))} d\mathcal{H}^\nu(x) = \int_{\varphi(D)} \sum_{x: x \in \varphi^{-1}(y)} f(x) d\mathcal{H}^\nu(y),$$

where $f : D \rightarrow \mathbb{E}$ (here \mathbb{E} is an arbitrary Banach space) is such that the function $f(x) \sqrt{\det(\widehat{D}\varphi(x)^* \widehat{D}\varphi(x))}$ is integrable, is valid. Here the Hausdorff measures are constructed with respect to metrics d_2 and \tilde{d}_2 with the multiple ω_ν .

Proof. 1ST STEP. Without loss of generality, we may assume that $D \subset \mathcal{U}$. In view of Theorem 3.10 we have $\mathcal{H}^\nu(\varphi(Z)) = 0$. It is left to prove the area formula for the set $A = D \setminus Z$. We may assume without loss of generality [1] that on the measurable set A we have

$$C_1 d_2(u, v) \leq \tilde{d}_2(\varphi(u), \varphi(v)) \leq C_2 d_2(u, v) \quad (3.1)$$

for some $0 < C_1, C_2 < \infty$, $\text{rank } \widehat{D}\varphi(z) = N$ for all points of hc -differentiability of the mapping φ , and the set A has the finite measure. For convenience, consider the case $f \equiv 1$. Note that the set function defined on open sets in $E \subset \mathbb{M}$,

$$\Phi(E) = \int_{\varphi(E \cap A)} d\mathcal{H}^\nu(y)$$

is absolutely continuous (since φ is a Lipschitz mapping: indeed, it is easy to see that there exists such $Q = Q(\varphi) < \infty$ that

$$\mathcal{H}^\nu(\varphi(E \cap D)) \leq Q \mathcal{H}^\nu(E \cap D)$$

for any set E) and additive. Consequently [1],

$$\Phi(A) = \int_A \Phi'(x) d\mathcal{H}^\nu(x).$$

Our goal is to show that

$$\Phi'(y) = \sqrt{\det(\widehat{D}\varphi(y)^* \widehat{D}\varphi(y))}$$

almost everywhere.

2ND STEP. For each $\varepsilon > 0$, there exists a set Σ_ε of the \mathcal{H}^ν -measure not exceeding ε , such that on $A \setminus \Sigma_\varepsilon$ the mapping φ is continuously hc -differentiable, i. e., the hc -differential $\widehat{D}\varphi(z)$, $z \in A \setminus \Sigma_\varepsilon$, is continuous [8, Lemma 4.6]. The definition of the hc -differentiability implies for $w, u \in A$ (here by our assumption u is a point of hc -differentiability of φ):

$$\widetilde{d}_2(\varphi(w), \varphi(u)) = \widetilde{d}_2(\widehat{D}\varphi(u)[w], \varphi(u)) + o(d_2(w, u)),$$

where $o(1) \rightarrow 0$ as $w \rightarrow u$. We also may assume without loss of generality that $o(\cdot)$ is uniform in $u \in A \setminus \Sigma_\varepsilon$. Since we have the assumption that $d_2(w, u) \geq \frac{1}{C_2} \widetilde{d}_2(\varphi(w), \varphi(u))$, it follows that

$$\widetilde{d}_2(\varphi(w), \varphi(u))(1 + o(1)) = \widetilde{d}_2(\widehat{D}\varphi(u)[w], \varphi(u)). \quad (3.2)$$

Here $o(1)$ is uniform in $u \in A \setminus \Sigma_\varepsilon$.

3RD STEP. Fix $\varepsilon > 0$ and prove the area formula for $A_\varepsilon = A \setminus \Sigma_\varepsilon$. Hereinafter in this proof, for the set $E \subset \mathbb{M}$, $E \subsetneq A_\varepsilon$, the symbol $\varphi(E)$ denotes $\varphi(E \cap A_\varepsilon)$. Fix $\sigma > 0$ and $r > 0$, and consider the set $\Delta_{\sigma r^\nu}$, $\mathcal{H}^\nu(\Delta_{\sigma r^\nu}) < \sigma r^\nu$, such that on $A_\varepsilon \setminus \Delta_{\sigma r^\nu}$, measurable functions

$$\Psi_m(y) = \frac{m^\nu}{\omega_\nu} \int_{\text{Box}_2(y, 1/m) \cap A_\varepsilon} d\mathcal{H}^\nu(x),$$

$m \in \mathbb{N}$, converge uniformly to the unity [1].

We may assume without loss of generality that $\Delta_{\sigma r^\nu} \subset \Delta_{\sigma t^\nu}$ for $r < t$. Indeed, it is sufficient to construct for each $l \in \mathbb{N}$ a set $\widetilde{\Delta}_l$, \mathcal{H}^ν -measure of which does not exceed $\sigma \left(\frac{1}{(l)^\nu} - \frac{1}{(l+1)^\nu} \right)$, and such that functions Ψ_m converge uniformly to the unity on $A_\varepsilon \setminus \widetilde{\Delta}_l$. Next, for $r \in (1/l, 1/(l-1)]$, put $\Delta_{\sigma r^\nu} = \bigcup_{k=l}^{\infty} \widetilde{\Delta}_k$; it is easy to see that its \mathcal{H}^ν -measure is not more than $\sigma/l^\nu < \sigma r^\nu$.

Moreover, for $r < t$ we have $\Delta_{\sigma r^\nu} \subset \Delta_{\sigma t^\nu}$. We will need this property at the end of the proof when $r, \sigma \rightarrow 0$ not to “loose” points we have considered.

Take $r > 0$ small enough, a density point $g \in A_\varepsilon \setminus \Delta_{\sigma r^\nu}$ of the set A_ε , and an open ball $\text{Box}_2(g, r)$. Since $\mathcal{H}^\nu(\Delta_{\sigma r^\nu}) \leq \sigma r^\nu$, it follows that $\mathcal{H}^\nu(\varphi(\Delta_{\sigma r^\nu})) \leq Q\sigma r^\nu$, and

$$\mathcal{H}^\nu(\varphi(\text{Box}_2(g, r))) \leq \mathcal{H}^\nu(\varphi(\text{Box}_2(g, r) \setminus \Delta_{\sigma r^\nu})) + Q\sigma r^\nu.$$

Fix $\tau > 0$ and (for fixed σ and r) choose such $\delta \leq \delta_0(\tau, \sigma, r)$, $\delta_0 \in (0, \sigma r)$, that for $1/m \leq \min\{\delta, \delta C_1\}$, where C_1 is taken from (3.1), we have $\Psi_m(y) \geq 1 - \tau$ for $y \in \text{Box}_2(g, r) \cap A_\varepsilon \setminus \Delta_{\sigma r^\nu}$ (it is possible in view of the uniform convergence of $\Psi_m(y)$ to the unity on $A_\varepsilon \setminus \Delta_{\sigma r^\nu}$).

For the chosen $\delta > 0$, construct the covering $\{\text{Box}_2(x_i, r_i)\}_{i \in \mathbb{N}}$ of the image $\varphi(\text{Box}_2(g, r) \setminus \Delta_{\sigma r^\nu})$ from the definition of \mathcal{H}_δ^ν .

The definition of the set $\Delta_{\sigma r^\nu}$ implies that for any covering by balls $\{\text{Box}_2(x_i, r_i)\}_{i \in \mathbb{N}}$ of the image $\varphi(\text{Box}_2(g, r) \setminus \Delta_{\sigma r^\nu})$ from the definition of \mathcal{H}^ν -measure, the centers x_i are images of the points $y_i \in \text{Box}_2(g, r) \cap A_\varepsilon \setminus \Delta_{\sigma r^\nu}$ that are density points of the set $\text{Box}_2(g, r) \cap A_\varepsilon$.

4TH STEP. To each point $y_i = \varphi^{-1}(x_i)$, assign the \mathcal{P} -differentiable mapping η_i of local Carnot groups defined as follows: $\mathcal{G}^{y_i} \mathbb{M} \ni w \xrightarrow{\eta_i} \widehat{D}\varphi(y_i)[w] \in \mathcal{G}^{\varphi(y_i)} \widetilde{\mathbb{M}}$. Each such mapping belongs to the class C^1 (in the classical sense), and it is contact (as a mapping of local Carnot groups) since $\eta_i(w) = \theta_{x_i} \circ L \circ \theta_{y_i}^{-1}(w)$. Here the linear mapping L is defined by the matrix of the hc -differential $\widehat{D}\varphi(y_i)$ in the following sense: first, the mapping $\theta_{y_i}^{-1}$ “calculates” the coordinates of w with respect to y_i , then the linear mappings L matrix of which coincides with the matrix of the hc -differential $\widehat{D}\varphi$ in the bases $\{\widehat{X}_j^{y_i}\}_{j=1}^N$ and $\{\widehat{X}_k^{x_i}\}_{k=1}^{\widetilde{N}}$ acts on the obtained point of \mathbb{R}^N , and finally, the mapping θ_{x_i} assigns a point on \mathbb{M} to this image point from \mathbb{R}^N . Recall that in view of the property

$$\widehat{D}\varphi(y_i)[X, Y] = [\widehat{D}\varphi(y_i)X, \widehat{D}\varphi(y_i)Y],$$

where X, Y are vector fields corresponding to the local Carnot group $\mathcal{G}^{y_i} \mathbb{M}$ in the definition of the hc -differential $\widehat{D}\varphi$, its matrix has block-diagonal structure in the bases $\{\widehat{X}_j^{y_i}\}_{j=1}^N$ and $\{\widehat{X}_k^{x_i}\}_{k=1}^{\widetilde{N}}$. Besides of this, η_i is continuously \mathcal{P} -differentiable (see Corollary 3.3; [9]), and $\widehat{D}\eta_i(v) = \widehat{D}\varphi(y_i)$ for all v close enough to y_i . Indeed, $d_2^{x_i}(\eta_i(w), \widehat{D}\varphi(y_i)[w]) = 0 = o(d_2^{y_i}(v, w))$.

Next, in the definition of the value

$$\mathcal{H}_\delta^\nu(\varphi(\text{Box}_2(g, r) \setminus \Delta_{\sigma r^\nu})),$$

to each ball $\text{Box}_2(x_i, r_i)$, there corresponds the summand $\omega_\nu r_i^\nu$. Fix $i \in \mathbb{N}$. In view of the area formula for Carnot groups [12] (see Theorem 3.11), for each mapping η_i , $i \in \mathbb{N}$, we have

$$\omega_\nu r_i^\nu = \sqrt{\det(\widehat{D}\varphi^*(y_i)\widehat{D}\varphi(y_i)) \cdot \widehat{\mathcal{H}}^\nu{}^{y_i}(\eta_i^{-1}(\text{Box}_2(x_i, r_i)))},$$

where the symbol $\widehat{\mathcal{H}}^\nu{}^{y_i}$ denotes \mathcal{H}^ν -measure in the local Carnot group $\mathcal{G}^{y_i}\mathbb{M}$ with respect to $d_2^{y_i}$.

Now, consider the sets $\eta_i^{-1}(\text{Box}_2(x_i, r_i))$ and $\varphi^{-1}(\text{Box}_2(x_i, r_i)) \cap A_\varepsilon$. Note that, under the mapping η_i , the preimage of an open set $\text{Box}_2(x_i, r_i)$ is also open, moreover, it has a boundary consisting of the finite number of surfaces of the class C^1 . In view of (3.2) (for $u = y_i$), all points of the set $\varphi^{-1}(\text{Box}_2(x_i, r_i)) \cap A_\varepsilon$ are contained in an $o(r_i)$ -neighborhood of the set $\eta_i^{-1}(\text{Box}_2(x_i, r_i))$.

Indeed, it follows from the fact that if $w \in \varphi^{-1}(\text{Box}_2(x_i, r_i)) \cap A_\varepsilon$ then

$$d_2(\varphi(w), x_i) = (1 + o(1))d_2(\eta_i(w), x_i), \quad (3.3)$$

and consequently $\eta_i(w) \in \text{Box}_2(x_i, r_i(1 + o(1)))$. Here $o(1)$ is uniform in all $i \in \mathbb{N}$ due to the choice of A_ε .

Besides of this, according to (3.2) (for $u = y_i$), all the points of the set A_ε , lying inside $\eta_i^{-1}(\text{Box}_2(x_i, r_i))$ and such that the distance to $\partial[\eta_i^{-1}(\text{Box}_2(x_i, r_i))]$ is more than $o(r_i)$, belong to the set $\varphi^{-1}(\text{Box}_2(x_i, r_i)) \cap A_\varepsilon$. Indeed, if

$$w \in \eta_i^{-1}(\text{Box}_2(x_i, r_i(1 - o(1)))) \text{ then } \widetilde{d}_2(\varphi(w), x_i) \leq r$$

for suitable values of $o(1)$ (see (3.3)). Here $o(1)$ is uniform in all $i \in \mathbb{N}$.

Since $y_i \in \text{Box}_2(g, r) \cap A_\varepsilon \setminus \Delta_{\sigma r^\nu}$, we have

$$\begin{aligned} & \mathcal{H}^\nu(\eta_i^{-1}(\text{Box}_2(x_i, r_i)))(1 + o(1)) \\ & \geq \mathcal{H}^\nu(\varphi^{-1}(\text{Box}_2(x_i, r_i)) \cap A_\varepsilon) \geq \mathcal{H}^\nu(\eta_i^{-1}[\text{Box}_2(x_i, r_i(1 - o(1))]) \cap A_\varepsilon) \\ & \geq (1 - o(1))\mathcal{H}^\nu(\eta_i^{-1}(\text{Box}_2(x_i, r_i))) - \tau(1 + o(1))\mathcal{H}^\nu(\text{Box}_2(x_i, r_i/C_1)), \end{aligned} \quad (3.4)$$

where $o(1) \rightarrow 0$ as $r_i \rightarrow 0$ uniformly in all x_i , $i \in \mathbb{N}$, by the choice of $\delta > 0$.

5TH STEP. Theorem 3.11, the equalities $\widehat{D}\eta_i \equiv \widehat{D}\varphi(y_i)$, $i \in \mathbb{N}$, and the continuity of the hc -differential $\widehat{D}\varphi$ imply

$$\begin{aligned} \sum_{i \in \mathbb{N}} \omega_\nu r_i^\nu &= \sum_{i \in \mathbb{N}} \sqrt{\det(\widehat{D}\varphi^*(y_i)\widehat{D}\varphi(y_i)) \cdot \widehat{\mathcal{H}}^\nu{}^{y_i}(\eta_i^{-1}(\text{Box}_2(x_i, r_i)))} \\ &= \left(\sqrt{\det(\widehat{D}\varphi^*(g)\widehat{D}\varphi(g))(1 + o(1))} \right) \cdot \sum_{i \in \mathbb{N}} \widehat{\mathcal{H}}^\nu{}^{y_i}(\eta_i^{-1}(\text{Box}_2(x_i, r_i))), \end{aligned} \quad (3.5)$$

where $o(1) \rightarrow 0$ as $r \rightarrow 0$. Thus, the sum $\sum_{i \in \mathbb{N}} \omega_\nu r_i^\nu$ is close to the minimal value if and only if the sum $\sum_{i \in \mathbb{N}} \widehat{\mathcal{H}}^\nu{}^{y_i}(\eta_i^{-1}(\text{Box}_2(x_i, r_i)))$ is also close to its minimal value. Since on \mathbb{M} we have $\mathcal{H}^\nu = \widehat{\mathcal{H}}^\nu{}^{y_i}(1 + o(1))$ where $o(1) \rightarrow 0$ as the points of a measured set converge to y_i (it is enough to consider their expressions via Riemannian measures), $i \in \mathbb{N}$, we may consider the sum $\sum_{i \in \mathbb{N}} \mathcal{H}^\nu(\eta_i^{-1}(\text{Box}_2(x_i, r_i)))$ instead of $\sum_{i \in \mathbb{N}} \widehat{\mathcal{H}}^\nu{}^{y_i}(\eta_i^{-1}(\text{Box}_2(x_i, r_i)))$. Now, we calculate this value. Since the “balls”

$$\begin{aligned} & \{\eta^{-1}(\text{Box}_2(x, t)) : x = \varphi(y), y \in \text{Box}_2(g, r) \cap A_\varepsilon \setminus \Delta_{\sigma r^\nu}, \\ & \eta(w) = \widehat{D}\varphi(y)[w], t \in (0, \min\{\delta, \delta C_1\}), \eta^{-1}(\text{Box}_2(x, t)) \subset \text{Box}_2(g, r)\} \end{aligned}$$

have the doubling condition (with respect to the measure \mathcal{H}^ν in view of the relation $\mathcal{H}^\nu = \widehat{\mathcal{H}}^\nu{}^{y_i}(1 + o(1))$, see above), then Vitali Covering Theorem implies the existence of the collection $\{\eta_i^{-1}(\text{Box}_2(x_i, r_i))\}_{i \in \mathbb{N}}$, covering the set $\text{Box}_2(g, r) \cap A_\varepsilon \setminus \Delta_{\sigma r^\nu}$ up to a set of \mathcal{H}^ν -measure zero. For this (remaining) set, there exists an at most countable covering by “balls” $\{\eta_j^{-1}(\text{Box}_2(x_j, t_j))\}_{j \in \mathbb{N}}$, with the sum of their \mathcal{H}^ν -measures less than σr^ν .

Relation (3.2) implies that $\bigcup_{i \in \mathbb{N}} \text{Box}_2(x_i, \tilde{r}_i) \cup \bigcup_{j \in \mathbb{N}} \text{Box}_2(x_j, \tilde{t}_j) \supset \varphi(\text{Box}_2(g, r) \setminus \Delta_{\sigma r^\nu})$, where $\tilde{r}_i = r_i(1 + o(1))$ and $\tilde{t}_j = t_j(1 + o(1))$, and $o(1)$ are uniform in all i, j . Moreover, the sum S of \mathcal{H}^ν -measures of the preimages of these balls under the corresponding mappings η_i can be estimated as

$$[\mathcal{H}^\nu(\text{Box}_2(g, r)) + \sigma r^\nu](1 + o(1)) \geq S \geq \mathcal{H}^\nu(\text{Box}_2(g, r) \cap A_\varepsilon \setminus \Delta_{\sigma r^\nu}),$$

where $o(1) \rightarrow 0$ as $r_i, t_j \rightarrow 0$, $i, j \in \mathbb{N}$. In view of (3.4), we have

$$\begin{aligned} S \cdot (1 + o(1)) & \geq \sum_{i \in \mathbb{N}} \mathcal{H}^\nu(\varphi^{-1}(\text{Box}_2(x_i, r_i)) \cap A_\varepsilon) \\ & + \sum_{j \in \mathbb{N}} \mathcal{H}^\nu(\varphi^{-1}(\text{Box}_2(x_j, t_j)) \cap A_\varepsilon) \geq (1 - o(1) - O(\tau))S, \end{aligned} \quad (3.6)$$

where $o(1) \rightarrow 0$ as $\delta \rightarrow 0$, and $O(1)$ is bounded uniformly on A_ε .

Note that, the sum $\sum_{k \in \mathbb{N}} \mathcal{H}^\nu(\varphi^{-1}(\text{Box}_2(x_k, r_k)) \cap A_\varepsilon)$ cannot be less than $\mathcal{H}^\nu(\text{Box}_2(g, r) \cap A_\varepsilon \setminus \Delta_{\sigma r^\nu})$ in case of any covering of $\varphi(\text{Box}_2(g, r) \setminus \Delta_{\sigma r^\nu})$ by any collection $\{\text{Box}_2(x_k, r_k)\}_{k \in \mathbb{N}}$. Consequently, we have for the value (see

(3.6))

$$\begin{aligned} \frac{\mathcal{H}^\nu(\text{Box}_2(g, r) \cap A_\varepsilon \setminus \Delta_{\sigma r^\nu})}{1 + o(1)} &\leq S \\ &\leq \frac{1}{1 - o(1) - O(\tau)} [\mathcal{H}^\nu(\text{Box}_2(g, r) \cap A_\varepsilon \setminus \Delta_{\sigma r^\nu}) + O(1)\sigma r^\nu], \end{aligned}$$

where the values $O(1)$ are bounded uniformly in all small $\delta > 0$ and small $r > 0$, and $o(1) \rightarrow 0$ as $\delta \rightarrow 0$, is indeed close to the minimal one.

Since we have $\tau \rightarrow 0$ and $o(1) \rightarrow 0$ while $\delta \rightarrow 0$, and while $r \rightarrow 0$ we can take $\sigma = \sigma(r) \rightarrow 0$, then, taking into account the fact that g is the density point of the set A_ε , and $\mathcal{H}^\nu(\Delta_{\sigma r^\nu}) = o(r^\nu)$, where $o(1) \rightarrow 0$ as $r \rightarrow 0$, we deduce from (3.5) that

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^\nu(\varphi(\text{Box}_2(g, r) \cap A_\varepsilon))}{\mathcal{H}^\nu(\varphi(\text{Box}_2(g, r) \cap A_\varepsilon \setminus \Delta_{\sigma r^\nu}))} = 1 \text{ and } \lim_{r \rightarrow 0} \frac{S}{\omega_\nu r^\nu} = 1.$$

Consequently, (3.4) implies

$$\Phi'(g) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^\nu(\varphi(\text{Box}_2(g, r) \cap A_\varepsilon))}{\omega_\nu r^\nu} = \sqrt{\det(\widehat{D}\varphi^*(g)\widehat{D}\varphi(g))}.$$

Since the latter is valid for almost all $g \in A_\varepsilon$, it implies the area formula for the set A_ε . We use standard argument to derive the area formula for the set A . The theorem follows.

Remark 3.13. All results of this paper are also true for mappings of Carnot manifolds enjoying conditions from [18, Remark 2.2.19] with basis vector fields on \mathbb{M} belonging to $C^{1,\alpha}$, $\alpha > 0$, and basis vector fields on $\widetilde{\mathbb{M}}$ belonging to $C^{1,\widetilde{\alpha}}$, $\widetilde{\alpha} > 0$.

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References

- [1] *Federer H.* Geometric Measure Theory. NY: Springer, 1969.

- [2] *Hajlasz P.* Sobolev Mappings, Co-Area Formula and Related Topics // Proceedings on Analysis and Geometry (Edt.: S. K. Vodopyanov), Sobolev Institute Press, Novosibirsk, 2000. P. 227–254.
- [3] *Kirchheim B.* Rectifiable metric spaces: local structure and regularity of the Hausdorff measure // Proc. AMS, 1994. V. 121. P. 113–123.
- [4] *Ambrosio L., Kirchheim B.* // Rectifiable sets in metric and Banach spaces // Math. Ann., 2000. V. 318. P. 527–555.
- [5] *Karmanova M. B.* Metric Differentiability of Mappings and Geometric Measure Theory. // Doklady Mathematics, 2005. V. 71 (2). P. 224–227.
- [6] *Karmanova M.* Geometric Measure Theory Formulas on Rectifiable Metric Spaces // Contemporary Mathematics, 2007. V. 424. In: “The Interaction of Analysis and Geometry”. P. 103–137.
- [7] *Karmanova M. B.* Area and coarea formulas for the mappings of Sobolev classes with values in a metric space // Sib. Math. J., 2007. V. 48 (4). P. 778–788.
- [8] *Vodopyanov S.* Geometry of Carnot–Carathéodory Spaces and Differentiability of Mappings // Contemporary Mathematics, 2007. V. 424. In: “The Interaction of Analysis and Geometry”. P. 247–301.
- [9] *Pansu P.* Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un. Ann. Math. (2), 1989. V. 129 (1). P. 1–60.
- [10] *Pauls S. D.* A notion of rectifiability modeled on Carnot groups // Indiana University Mathematics Journal, 2004. V. 53. P. 49–82.
- [11] *Magnani V.* Differentiability and Area formula on Stratified Lie groups // Houston J. Math., 2001. V. 27. P. 297–323.
- [12] *Vodopyanov S. K., Karmanova M. B.* An Area Formula for Contact C^1 -Mappings of Carnot Manifolds // Complex Variables and Elliptic Equations, 2009 (accepted).
- [13] *Karmamova M. B.* A New Approach to Investigation of Carnot–Carathéodory Geometry. // Doklady Mathematics, 2010. V. 82, No 2. P. 746–750.

- [14] *Karmanova M.* The New Approach to investigation of Carnot–Carathéodory Geometry. // GAFA, 2011, accepted.
- [15] *Varchenko A. N.* On Obstructions to the Local Equivalence of Distributions // Mat. Zametki, 1981. V. 29 (6). P. 939–947.
- [16] *Gromov M.* Carnot–Carathéodory Spaces Seen From Within // In: Sub-Riemannian geometry. Basel, Birkhäuser Verlag, 1996. P. 79–318.
- [17] *Nagel A., Stein E. M., Wainger S.* Balls and metrics defined by vector fields I: Basic properties. Acta Math., 1985. V. 155. P. 103–147.
- [18] *Vodopyanov S., Karmanova M.* Geometry of Carnot–Carathéodory Spaces, Differentiability, Coarea and Area Formulas // In: “Analysis and Mathematical Physics”, Birkhäuser, 2009. P. 233–335.
- [19] *Mitchell J.* // On Carnot–Carathéodory metrics. J. Differential Geometry, 1985. V. 21. P. 35–45.
- [20] *Vodopyanov S. K., Karmanova M. B.* Local Geometry of Carnot Manifolds Under Minimal Smoothness. // Doklady Mathematics, 2007. V. 75 (2). P. 240–246.
- [21] *M. M. Postnikov, Lectures in Geometry. Semester V: Lie Groups and Lie Algebras.* Moscow, ”Nauka“, 1982.
- [22] *A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, Stratified Lie Groups and Potential Theory for Their Sub-Laplacians.* Springer, 2007.
- [23] *G. B. Folland, E. M. Stein, Hardy spaces on homogeneous groups.* Princeton Univ. Press, 1982.
- [24] *Vodopyanov S. K.* The theory of Lebesgue integral: *lecture notes in calculus* // Novosibirsk State University Press, Novosibirsk, 2006.